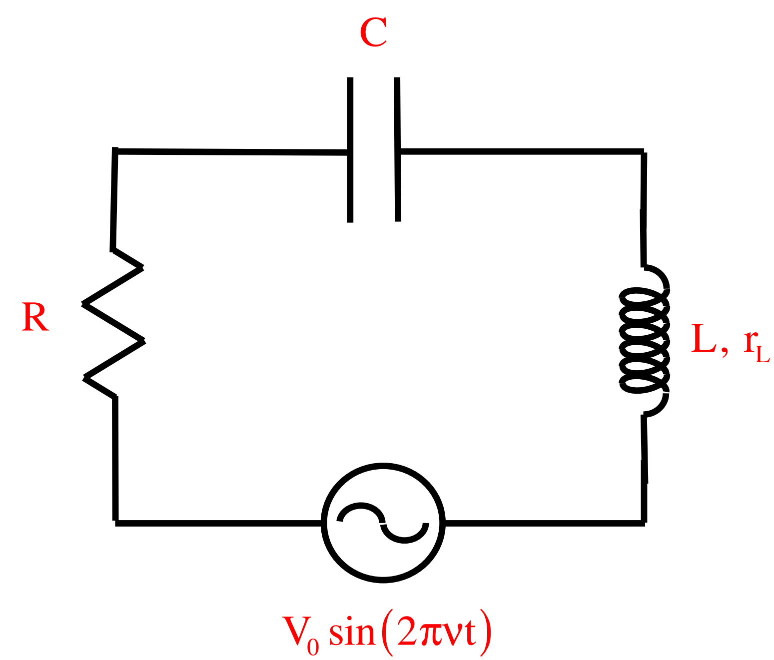
RLC circuits - finally!

So what happens when we have a resistor, capacitor, and inductor all in the same circuit (and RLC circuit)?



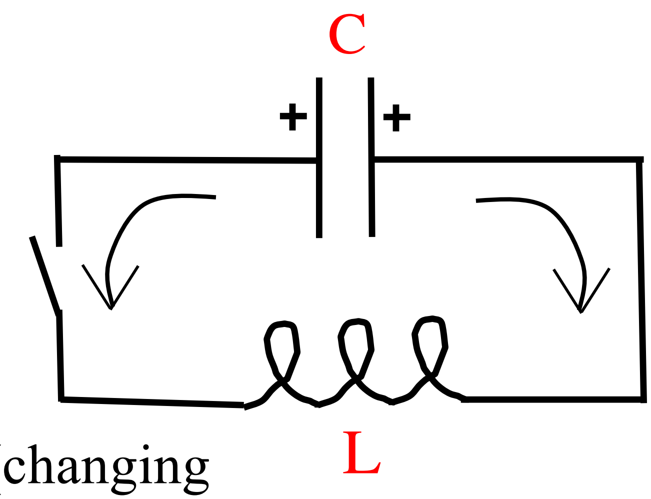
All hell breaks loose:

The inductor will try to make the voltage *lag* the current and fights to suppress the AC source signal if it happens to be high frequency;

The capacitor will try to make the voltage *lead* the current and fights to suppress the AC source signal if it happens to be low frequency;

So you'd expect there would be NO frequency that would produce current in the circuit . . . except that's not the case. There will be one frequency at which the lead/lag characteristics of the inductor and capacitor will nullify one another leaving only the resistor-like resistance in the circuit to limit current.

But before we look at the math, consider the following situation conceptually:



1.) Consider a charged capacitor and an inductor in a circuit with very little resistor-like resistance.

2.) When the switch is thrown, the capacitor begins to discharge, attempting to send current through the inductor;

3.) The inductor responds by producing a back EMF (changing magnetic flux, after all) that fights the increase in charge flow.

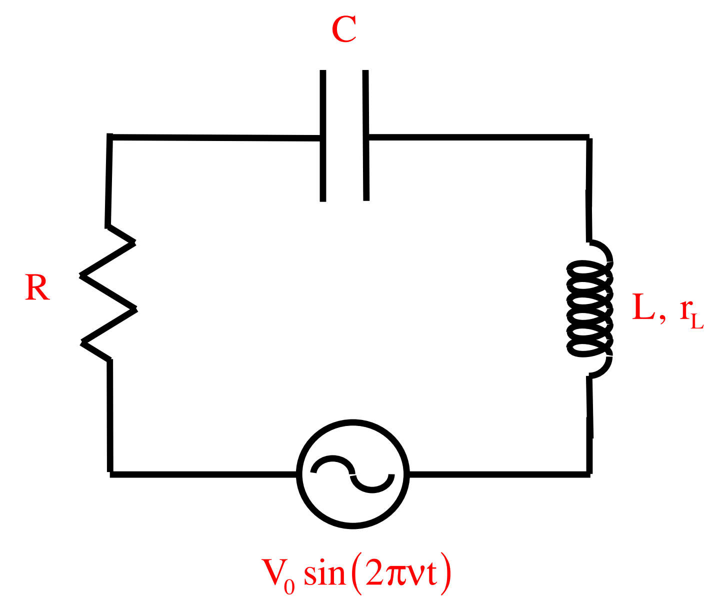
4.) As the cap's charge diminishes and the current slows, the inductor again fights the change by making charge flow MORE than it would have; by the time the cap runs out of charge the inductor will still be forcing current to flow.

5.) This will begin to charge up the other plate.

6.) Once the recharging stops, the cap will begin to discharge going the other way, and the cycle will replay itself. With no resistance, this will continue indefinitely.

7.) Bottom line: Once the process starts, current will oscillate in the circuit at some characteristic frequency, called the resonance frequency; that frequency will be dependent upon the size of the cap and inductor in the circuit.

There are two ways to go about looking at this. The first has to do with what are called phasor diagrams.



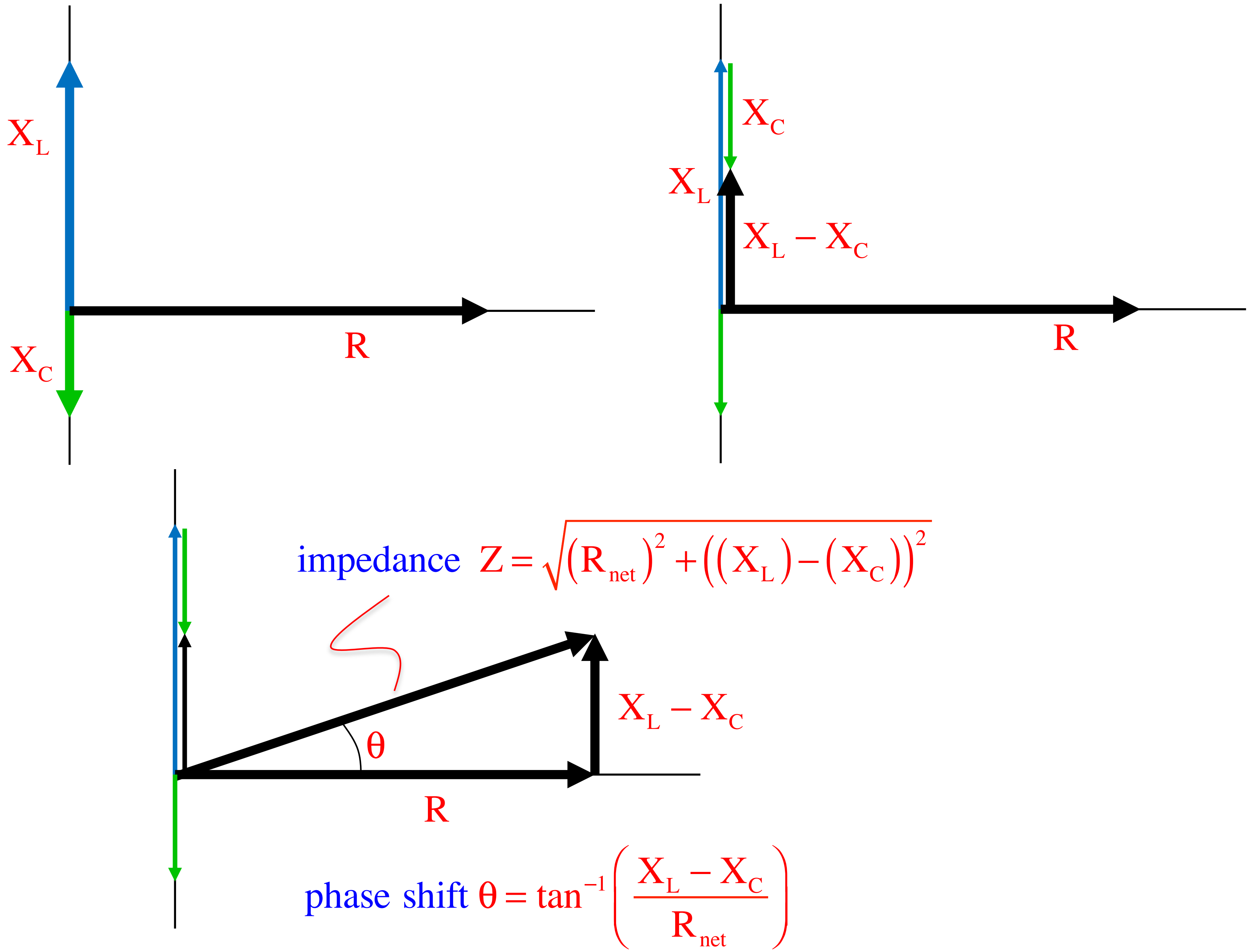
--In a phasor diagram, the resistive nature of all of the elements that don't throw the voltage out of phase with the current are graphed along the *x axis*;

--The resistive nature of all of the elements that make the voltage lead the current by a quarter of a cycle are graphed along the +*y axis*;

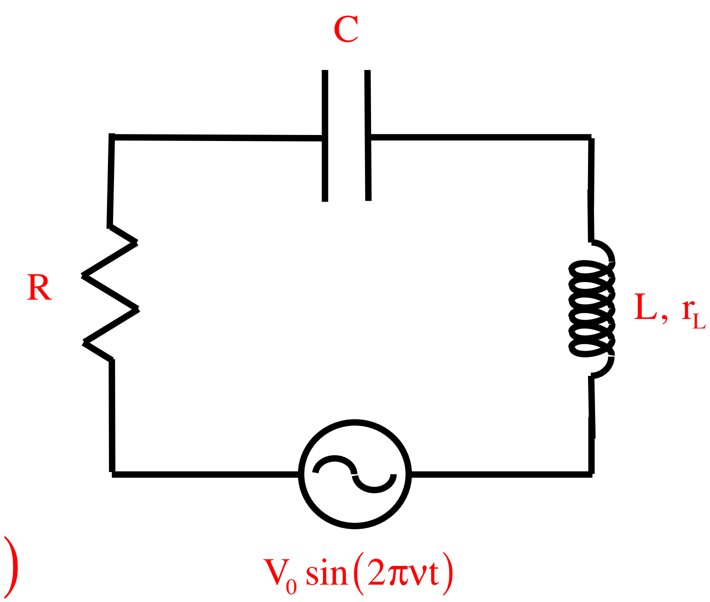
--The resistive nature of all of the elements that make the voltage lag the current by a quarter of a cycle are graphed along the -*y axis*;

--Once done, the vectors are added . . . (which is all happening on the next page)

Phasor diagram for an RLC circuit



To the math: Kirchoff's loop equation yields:



$$-L\frac{di}{dt} + V_o \sin(2\pi\nu t) - i(R + r_L) - \frac{q}{C} = 0$$

$$\Rightarrow \quad \frac{di}{dt} + \frac{(R + r_L)}{L} i + \frac{q}{LC} = \frac{V_o}{L}\sin(2\pi\nu t)$$

$$\Rightarrow \quad \frac{dq^2}{d^2t} + \frac{(R + r_L)}{L}\frac{dq}{dt} + \left(\frac{1}{LC}\right) q = \frac{V_o}{L}\sin(2\pi\nu t)$$

5.) Solving this second-order differential equation in q (which is a trip unto itself) will produce a solution for i whose denominator looks like:

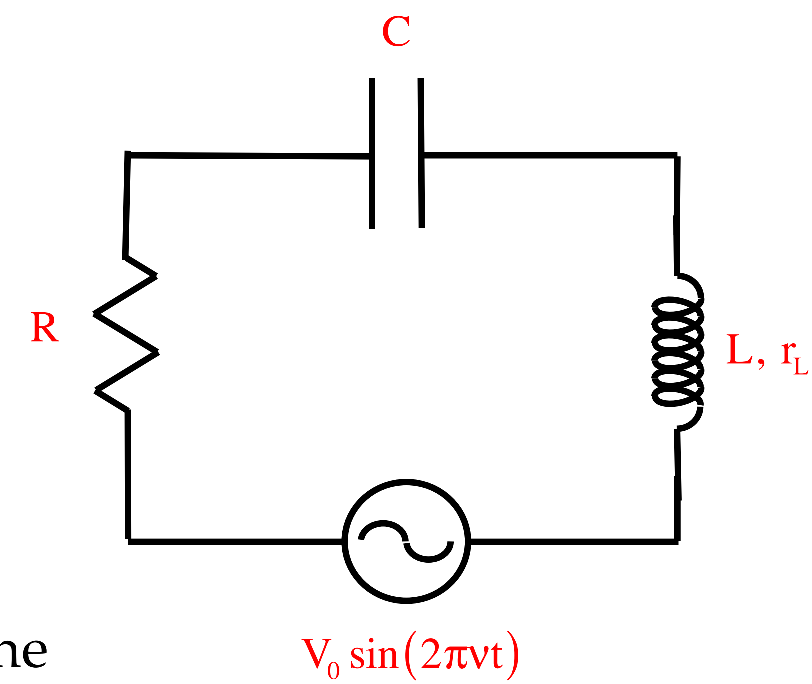
$$i = \frac{\text{voltage term}}{\text{net resistive nature}}$$

$$\left[(R + r_L)^2 + \left(2\pi\nu L - \frac{1}{2\pi\nu C}\right)^2\right]^{1/2}$$

6.) This overall "frequency dependent" resistive nature for the RLC circuit, which is sometimes written as $Z = \left[(R_{net})^2 + \left((X_L)^2 - (X_C)^2\right)\right]^{1/2}$,

is called the circuit's impedance and is given (as shown) the symbol Z.

7.) Things to notice about this relationship.



a.) Ohm's Law works just fine in these circuits, except now it is written as $V = i\,Z$, where the impedance Z is the net resistive nature of the circuit.

b.) With no capacitor or inductor in the circuit, the impedance simply becomes the resistance R in the circuit. That means $V = iR$ Is really just a special case of this more expanded version.

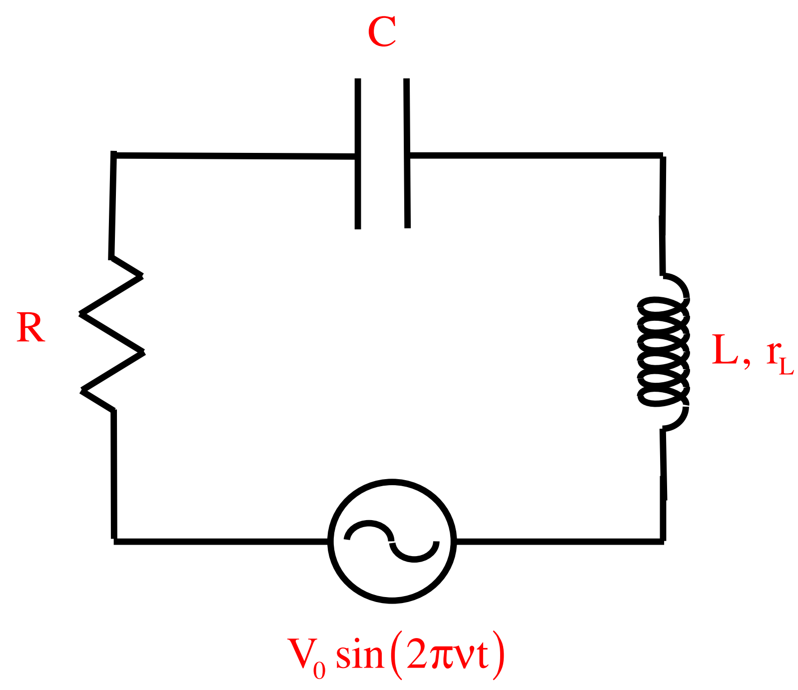
c.) There is a frequency at which the impedance is a minimum and the current is a maximum. It happens when the $\left(2\pi\nu L - \frac{1}{2\pi\nu C}\right)$ part goes to zero.

That is, when $2\pi\nu L = \frac{1}{2\pi\nu C}$, or when:

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

This is that resonance frequency at which the circuit's current naturally wants to oscillate.

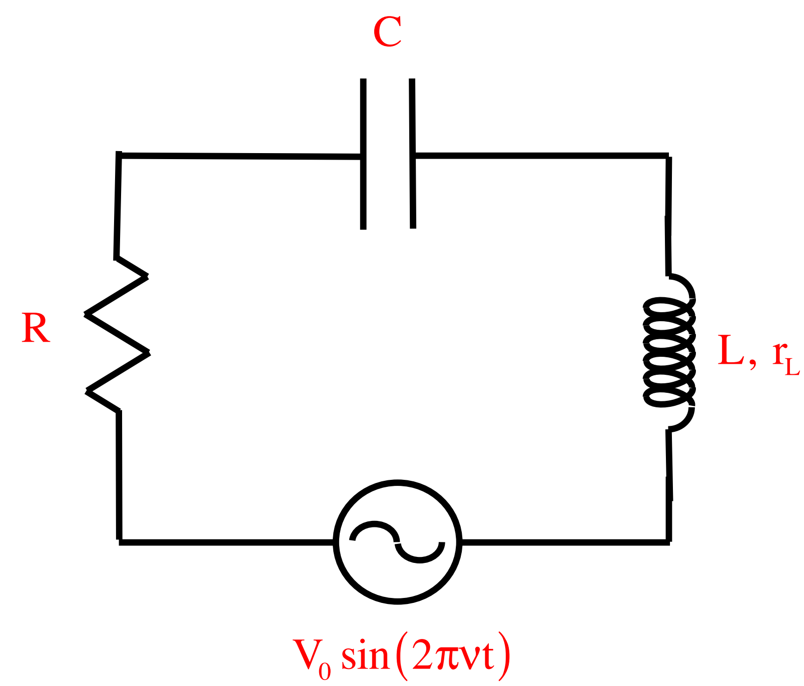
d.) At resonance frequency, the only resistance in the circuit is resistor-like resistance and that signal proliferates.



i.) What this means is that the frequency response curve for a typical RLC circuit looks like:

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

e.) This frequency-response characteristic is going to be very important in the radio circuit (more about this later).



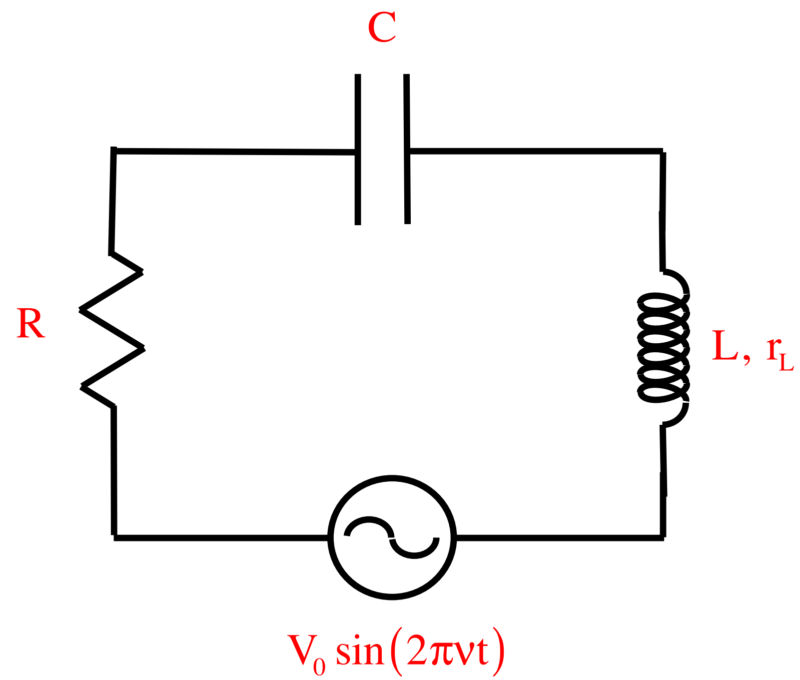
f.) And as an interesting note: you now know what the impedance label on the back of your stereo speakers means. It is telling the you speaker's net, overall resistance to charge flow due to all of the resistors, capacitors or inductors in the speaker circuit when the frequency is between 250 Hz and 400 Hz.

In other words, when a speaker says, Impedance: 8 ohms, it means that for the circuit when run at, say, 300 Hz,

$$Z = \left[\left(R + r_L\right)^2 + \left(2\pi\nu L - \frac{1}{2\pi\nu C}\right)^2 \right]^{1/2} = 8\ \Omega.$$

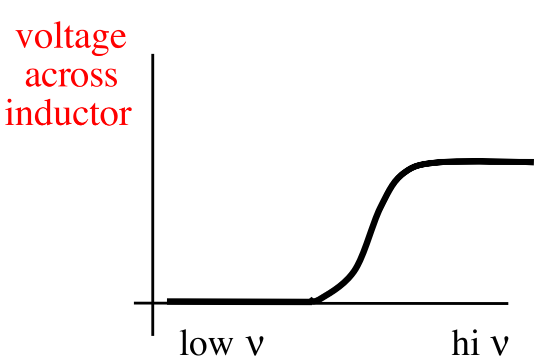
8.) Lastly, it should be noted that the calculation of the phase shift

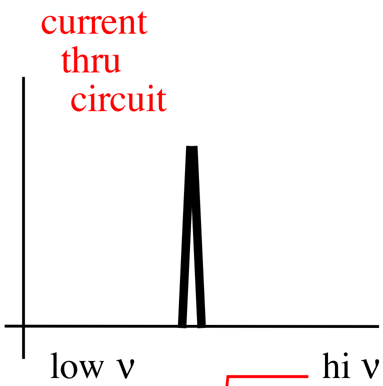
$$\theta = \tan^{-1}\left(\frac{X_L - X_C}{R_{net}}\right)$$

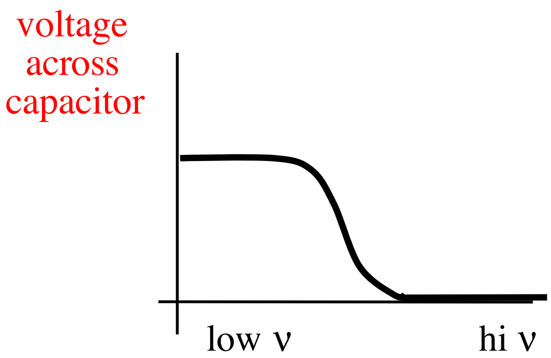


--whether the voltage leads or lags the current--identifies which is happening by its sign. If the angle is positive, it means the voltage leads the current; if the angle is negative, it means the voltage lags the current.

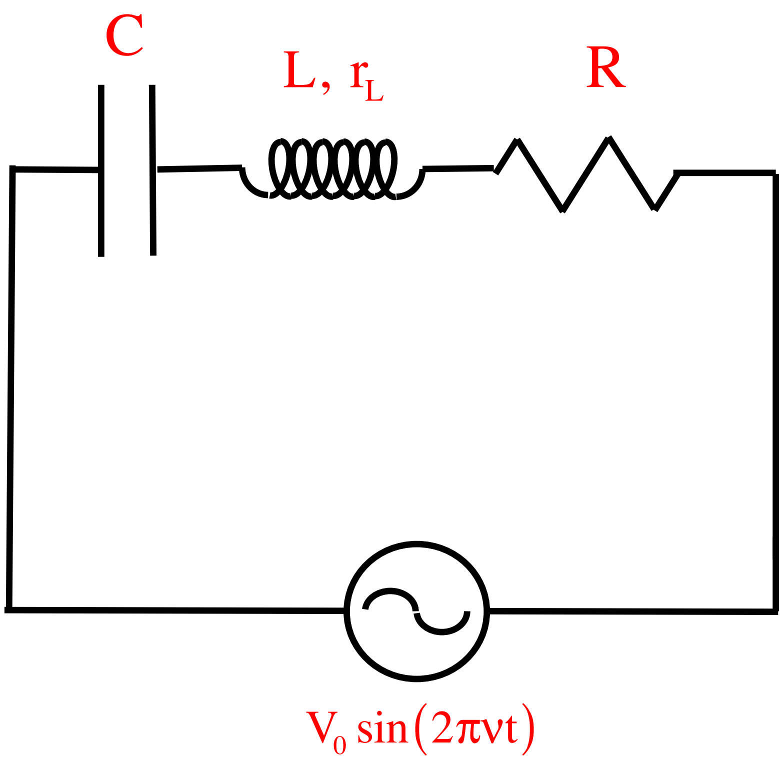
RLC circuit: As we pan through the frequencies:

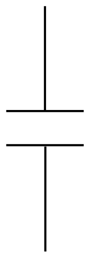
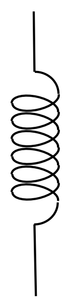






$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$



element	symbol	units	resistive nature	Filter?	phase relationship
Resistor	 R	ohms	“resistor-like” resistance R (ohms)	no	current in phase with voltage across element
Capacitor	 C	farads	at low ν, V_c big, V_R low $\Rightarrow$ low i so cap's frequ. dependent res. nature is big *capacitive reactance*—frequency dep. resistive nature of in an RC circuit: $X_C = \frac{1}{2\pi\nu C}$ ohms	high pass	with minimal resistance-like resistance in circuit, voltage LAGS current by $\frac{\pi}{2}$ radians
Inductor	 L	henrys	at low ν, V_L small, V_R big $\Rightarrow$ big i so ind's frequ. dependent res. nature is small *inductive reactance*—frequency dep. resistive nature of in an RL circuit: $X_L = 2\pi\nu L$ ohms also, “resistor-like” resistance r_L	low pass	with minimal resistance-like resistance in circuit, voltage LEADS current by $\frac{\pi}{2}$ radians
RLC ckt.	Z	ohms	***Impedance = total resistive nature of circuit*** $Z = \left[(R + r_L)^2 + \left(2\pi\nu L - \frac{1}{2\pi\nu C} \right)^2 \right]^{1/2}$ Minimum Z, max i at resonance frequ. when $\nu_{resonance} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$	passes res. frequ.	phase shift defined by: $\theta = \tan^{-1}\left(\frac{X_L - X_C}{R_{net}} \right)$ V leads i if $\theta > 0$, V leads i if $\theta < 0$, V lags i